Observations on pointing the HET

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Summary

This document contains two discussions. Section 1 discusses two corrections for the computation of the parallactic angle and for the rotation of the field of view. Section 2 discusses two possible transformations to orient the hexapod, and a correction for the transformation currently performed at the HET. Note that we prefer to orient the hexapod using "Frank Ray's transformation" (Section 2.3).

1 Corrections for angles in the focal plane

1.1 Parallactic angle as seen on the focal plane

The parallactic angle is the angle p formed by the transit plane and the YZ plane of the ITF (see [Ray1], Figure 1). The transit plane contains the axis Z of the ITF, hence the projection of the transit plane and the YZ plane onto the YX plane are two lines which form an angle p. In other words, looking down the Z axis of the ITF we see exactly the parallactic angle (see [Ray2], Figure 11).

Let us call the **focal plane** the plane perpendicular to the w axis of the SIRP. The transit plane intersects the focal plane on a line a, which is the line containing the north pole N that we see on the field of view. The YZ plane intersects the focal plane on a line b, which is the line one sees as the "up" direction on the focal plane.

Figure 1 tries to illustrate the general situation. The two vertical planes (purple) represent the YZ plane and the transit plane. The horizontal plane (cyan) represents the XY plane. The slanted plane represents the focal plane. The intersection of the horizontal and vertical planes (bold black lines) is the parallactic angle p, while the intersection of the slanted and vertical planes (bold blue lines) is the angle $\tilde{p}$ one "sees" on the focal plane.

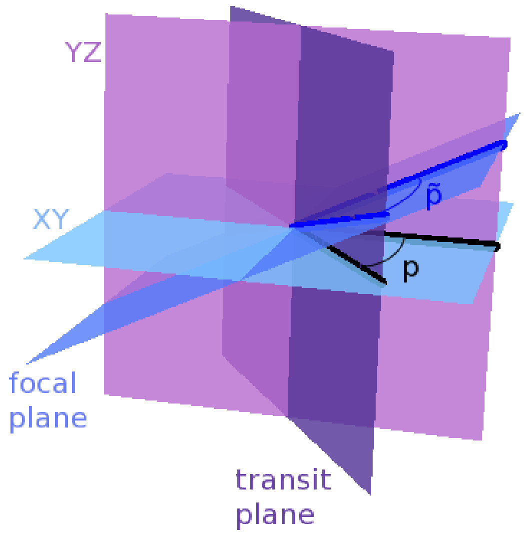

Figure 1: $\tilde{p}$ is the parallactic angle as seen on the focal plane.

The angle in blue is given by

$$\tilde{p} = \arccos\left(\frac{\sin p \sin\phi \sin\theta \cos\theta + \cos p \cos\phi}{\sqrt{\sin p \cos\theta(\cos p \sin(2\phi)\sin\theta + \sin p \cos\theta) + \cos^2 p \cos^2\phi}}\right).$$

When $\phi = \theta = 0$ (when the line of sight coincides with the Z axis of the ITF) the previous equation reduces to $\tilde{p} = p$.

However, for other values of ϕ and θ, the angle $\tilde{p}$ is slightly different than p. For example, for $\phi \approx \theta \approx 6.35°$ (so that $\beta \approx 8.8°$), we have

$$\tilde{p} = \arccos\left(\frac{0.0121627 \sin p + 0.993862 \cos p}{\sqrt{0.987762 \cos^2 p + 0.993862 \sin p(0.993862 \sin p + 0.0243253 \cos p)}}\right).$$

Plotting the expression $p - \tilde{p}$ as a function of p we get the graph shown in Figure 2. Both axes are in degrees.

1.2 Rotation of the sky as seen on the focal plane

In [Ray1], the rotation necessary to compensate the rotation of the sky is computed as $\rho = -\arcsin\big(\sin\delta(t)\sin h_C(t)\big)$, where δ is the declination and h_C is the hour angle with

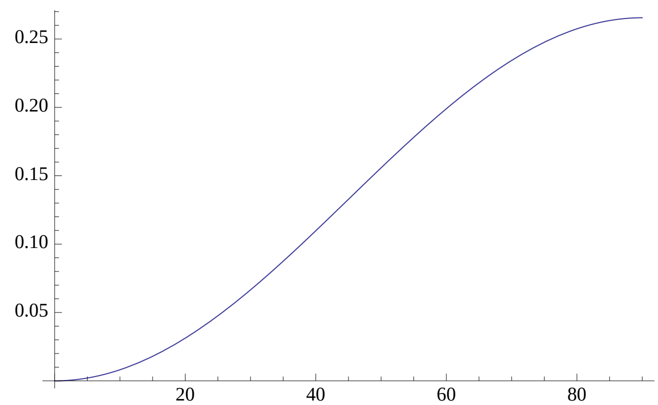

Figure 2: Graph of $p - \tilde{p}$ as a function of p.

respect to the transit plane. This value is measuring the angle between a line in the focal plane pointing north and its orthogonal projection on the transit plane.

In Figure 3 we have a picture of the celestial sphere. The blue circle is a line of constant declination, the green arcs are the hour angle h_C and the declination δ, the bold black line is the line of sight, the rightmost red line is the direction of north in the focal plane, and the leftmost red line is its projection onto the transit plane. The angle ρ measures the angle between the two red lines.

However, the plane defined by the two red lines does not coincide with the focal plane in general (unless $\rho = 0$, or when $\delta(t) = \pi/2$), so the angle ρ seen on the focal plane differs slightly.

In the focal plane we see the angle formed by the rightmost red line and the intersection of the focal plane and the transit plane. We can derive this angle as follows.

The line of sight is the vector

$$w = (\cos\delta\cos h_C, \cos\delta\sin h_C, \sin\delta),$$

since we are looking at a point with polar coordinates (h_C, δ). Consider the plane determined by the line of sight w and the pole (the line with direction $\overrightarrow{ON}$). Its normal is

$$n = w \times (0, 0, 1) = (\cos\delta\sin h_C, -\cos\delta\cos h_C, 0).$$

The north direction on the focal plane (the rightmost red line) is, then, the intersection of the plane with normal n and the focal plane (with normal w). Hence, its intersection is

$$a = n \times w = \cos\delta(-\sin\delta\cos h_C, -\sin\delta\sin h_C, \cos\delta).$$

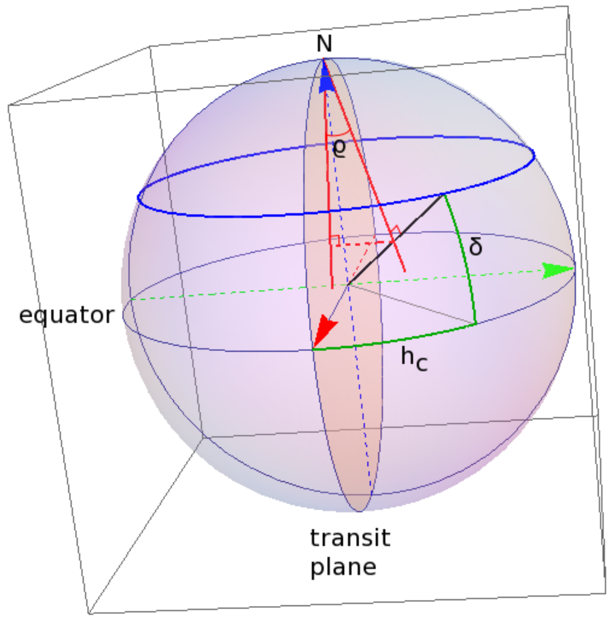

Figure 3: Angle ρ in the celestial sphere.

On the other hand, the intersection of the focal plane with the transit plane (which has normal $(0, 1, 0)$) is

$$b = w \times (0, 1, 0) = (-\sin\delta, 0, \cos\delta\cos h_C).$$

The angle between the vectors a and b is the angle ρ we want to measure:

$$\rho(t) = \arccos\left(\frac{a \cdot b}{\|a\|\|b\|}\right) = \arccos\left(\frac{\cos h_C(t)}{\sqrt{\cos^2\delta(t)\cos^2 h_C(t) + \sin^2\delta(t)}}\right), \tag{1}$$

and the sign of $\rho(t)$ must coincide with the sign of $-\delta(t)h_C(t)$. This formula is valid for $\delta(t)$ and $h_C(t)$ in the range $(-\pi/2, \pi/2)$.

We can verify the cases where this expression must coincide with Frank Ray's ρ:

- when we are on the transit plane ($h_C = 0$), then Equation (1) reduces to 0;
- when $\delta(t) = 0$, for any $h_C(t)$, the focal plane is parallel to the pole line, and Equation (1) correctly reduces to 0;
- when $\delta(t) = \pi/2$, the focal plane is perpendicular to the pole line, and $\rho(t)$ coincides with $h_C(t)$, as it does Frank Ray's ρ.

For the general case, the difference between Equation (1) and Frank Ray's ρ as a function of δ and h_C is shown in Figure 4 (all axes are in degrees).

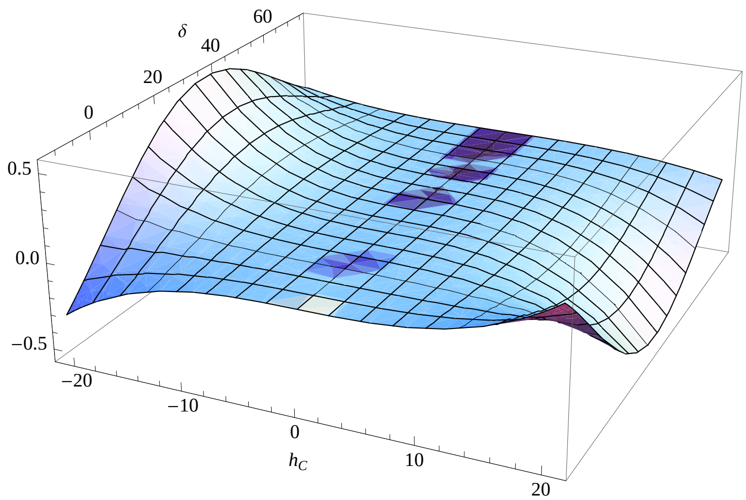

Figure 4: Graph of $\rho - \rho_{\mathrm{Ray}}$ as a function of δ and h_C.

2 Hexapod transformations

2.1 Rotation of the field of view for Jim's transformation

According to [Fow] the transformation used to orient the hexapod is

$$R_{p1} = R_Z\big(-(\rho+\zeta)\big)R_Y(\beta)R_Z(\zeta) \tag{2}$$

where

$$R_Z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\beta = \arctan\sqrt{\tan^2\phi + \tan^2\theta}, \quad \zeta = \operatorname{atan2}(\tan\theta, \tan\phi).$$

(we are ignoring the translations since we are only interested in angles, for this discussion). The selection of β and ζ is made so that the the projections of $w = R_{p1}Z$ onto the XZ and YZ form angles ϕ and θ with the Z axis, respectively, as required.

We can check the previous statement by computing those angles. For example, the angle between the Z axis and the projection of $R_{p1}(0,0,1)$ onto YZ is

$$\arccos\left(\frac{\cos\beta}{\sqrt{\cos^2\beta+\sin^2\beta\sin^2(\rho+\zeta)}}\right) \tag{3}$$

When $\rho = 0$, as is assumed in [Fow], the previous expression reduces to θ.

Note that equation (3) **does not** coincide with θ when $\rho \neq 0$. The value of ρ in [Fow] is interpreted as a possible extra rotation of the hexapod. To keep the right orientation for the normal to the top of the hexapod this extra rotation by ρ should be added to the the rightmost R_Z transformation (see equation (5) in Subsection 2.2) instead of the leftmost transformation as is done in equation (2).

To see the rotation introduced by the transformation R_{p1} (with $\rho = 0$) on the field of view, we want to compare the effect of the transformation on the Y axis (which will become the y axis on the SIRP), with the "up" direction on the focal plane (the intersection of the YZ plane and the focal plane).

Transforming the vector $(0,1,0)$ by R_{p1} and computing the angle with the line b from the previous section gives:

$$\rho_c = \arccos\big(\sin(\beta)\sin(\theta)\sin(\zeta)+\cos(\theta)\left(\cos(\beta)\sin^2(\zeta)+\cos^2(\zeta)\right)\big) \tag{4}$$

If we rotate the SIRP around w by $-\rho_c$, the y axis will be in the "up" direction, parallel to the YZ plane, and forming an angle $\tilde{p}$ (from previous section) with the north pole.

For example, when $\phi = \theta = 6°$, we get $\rho_c \approx 0.31°$.

2.2 Correction to Jim's transformation

Consider the transformation

$$T = T_{(X,Y,Z)} R_Z(-\zeta) R_Y(\beta) R_Z(\zeta + \rho_c) \tag{5}$$

where $T_{(X,Y,Z)}$ is a translation of vector (X,Y,Z), and ρ_c is defined as in (4).

This transformation maps the Z axis onto the w axis, so that it is tilted according to ϕ and θ, and keeps the transformation of the Y axis parallel to the original YZ plane.

2.3 Frank Ray's transformation

In [Ray1], Section 5, Frank Ray proposes a transformation to tilt the hexapod that keeps the transformation of the Y axis parallel to the YZ plane of the ITF. It is important to note that Ray's transformation in [Ray1] is written only for the "can on a string" situation. This means that the angles θ and ϕ depend on X and Y, in such a way that the w axis is always pointing to the center of curvature (which has ITF coordinates $(0, 0, F_S)$).

To write Ray's transformation for the general situation, where the variables X, Y, Z, θ, and ϕ do not depend on each other, let

$$\tilde{\phi} = \arccos\left(\frac{\cos\phi}{\sqrt{\cos^2\theta\sin^2\phi + \cos^2\phi}}\right).$$

The sign for $\tilde{\phi}$ must match the sign of ϕ. Then, the transformation

$$T = T_{(X,Y,Z)} R_X(-\theta) R_Y(\tilde{\phi}), \tag{6}$$

where

$$R_X(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

transform the Z axis into the w axis and keeps the transformed Y axis parallel to the YZ plane. This means that the transformations in (5) and (6) are exactly the same, as they are orthogonal transformations which coincide on two linearly independent vectors (Z axis and Y axis).

The particular case of "can on a string" occurs when X, Y, Z, θ, ϕ are related by the equations

$$\theta = \arctan\left(\frac{Y}{F_S - Z}\right), \quad \phi = \arctan\left(\frac{X}{F_S - Z}\right), \quad Z = F_S - \sqrt{F_S^2 - X^2 - Y^2}.$$

For these values, the rotations $R_X(-\theta)$ and $R_Y(\tilde{\phi})$ coincide with the rotations ($20a$) and (21) in [Ray1]. The simplifications in equation ($21a$) in [Ray1] contain typos; the correct matrices are:

$$R_X(-\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{F_S - Z}{RV_{YZ}} & \frac{-Y}{RV_{YZ}} & 0 \\ 0 & \frac{Y}{RV_{YZ}} & \frac{F_S - Z}{RV_{YZ}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_Y(\tilde{\phi}) = \begin{pmatrix} \frac{RV_{YZ}}{F_S} & 0 & \frac{-X}{F_S} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{X}{F_S} & 0 & \frac{RV_{YZ}}{F_S} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $RV_{YZ} = \sqrt{F_S^2 - X^2}$.

2.4 Change of basis between the ITF and the SIRP frames

The SIRP frame is an orthogonal coordinate frame with axes named x, y, and w, satisfying:

- its origin is located at the Stationary Image Rotation Point (SIRP);
- the w axis is perpendicular to the top of the hexapod;
- the y axis is parallel to the YZ plane of the ITF.

If the hexapod is tilted using the transformation T discussed in sections 2.2 and 2.3, then the SIRP frame can be seen as bolted to the top of the hexapod.

An instrument fixed to the top of the hexapod can be allowed to rotate around the w axis of the SIRP frame using the ρ stage. Let us call SIRP_ψ the coordinate frame attached to such an instrument, where ψ is the angle that measures the rotation of the SIRP_ψ frame with respect to the SIRP frame. With this notation, we have $\mathrm{SIRP} = \mathrm{SIRP}_0$.

2.4.1 SIRP_ψ to ITF

The SIRP_ψ frame is determined from the ITF by 6 values (see [MEC]):

- (X, Y, Z): the origin of the SIRP_ψ is translated by this vector from the origin of the ITF;
- θ: let w_{YZ} be the projection of the axis w of the SIRP_ψ onto the plane YZ of the ITF, then θ is the angle between w_{YZ} and the Z axis of the ITF;
- ϕ: let w_{XZ} be the projection of the axis w of the SIRP_ψ onto the plane XZ of the ITF, then ϕ is the angle between w_{XZ} and the Z axis of the ITF;
- ψ: let b be the intersection of the YZ plane of the ITF with the plane perpendicular to w, then ψ is the angle between the y axis of the SIRP_ψ and the line b. The angle ψ is positive in the counter-clockwise direction when looked down from $+\infty$ on the w axis of the SIRP_ψ.

Then, the matrix of change of basis from the SIRP_ψ to the ITF is ${}_{\mathrm{ITF}}M_{\mathrm{SIRP}_\psi}$:

$$\mathrm{coords}_{\mathrm{ITF}}(v) = {}_{\mathrm{ITF}}M_{\mathrm{SIRP}_\psi} \cdot \mathrm{coords}_{\mathrm{SIRP}_\psi}(v)$$

for any point v, and

$$ {}_{\text{ITF}}M_{\text{SIRP}_\psi} = T_{(X,Y,Z)} R_X(-\theta) R_Y \left(\arccos\left(\frac{\cos\phi}{\sqrt{\cos^2\theta \sin^2\phi + \cos^2\phi}} \right) \right) R_Z(\psi), $$

where

$$ R_Z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

In the usual case, to compensate for the rotation of the sky, we would set $\psi = \tilde{\rho}$, but it is important to note that the change of coordinates is valid for an arbitrary value of ψ.

2.4.2 dSIRP to SIRP$_\psi$

Let dSIRP be a coordinate frame slightly moved with respect to the SIRP$_\psi$. A correction changes the dSIRP frame back to the SIRP$_\psi$. The dSIRP frame is determined from the SIRP$_\psi$ by 6 values:

- (dx, dy, dw): the origin of the dSIRP frame is translated by this vector from the origin of the SIRP$_\psi$;
- tip: the dSIRP frame is first rotated by an angle tip, counter-clockwise, around the x axis of the SIRP$_\psi$;
- tilt: the dSIRP frame is then rotated by an angle tilt, counter-clockwise, around the y axis of the SIRP$_\psi$;
- dr: the dSIRP frame is finally rotated by an angle dr, counter-clockwise, around the w axis of the SIRP$_\psi$.

The matrix of change of basis between dSIRP and SIRP$_\psi$ is ${}_{\text{SIRP}_\psi}M_{\text{dSIRP}}$. This means that

$$ \text{coords}_{\text{SIRP}_\psi}(v) = {}_{\text{SIRP}_\psi}M_{\text{dSIRP}} \cdot \text{coords}_{\text{dSIRP}}(v) $$

for any point v. Let

$$R_x(\mathrm{tip}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\mathrm{tip}) & -\sin(\mathrm{tip}) & 0 \\ 0 & \sin(\mathrm{tip}) & \cos(\mathrm{tip}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_y(\mathrm{tilt}) = \begin{pmatrix} \cos(\mathrm{tilt}) & 0 & \sin(\mathrm{tilt}) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\mathrm{tilt}) & 0 & \cos(\mathrm{tilt}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_z(dr) = \begin{pmatrix} \cos(dr) & -\sin(dr) & 0 & 0 \\ \sin(dr) & \cos(dr) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dw \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, ${}_{\mathrm{SIRP}_\psi}M_{\mathrm{dSIRP}} = TR_{z,dr}R_{y,\mathrm{tilt}}R_{x,\mathrm{tip}}$. The explicit expression is

$${}_{\mathrm{SIRP}_\psi}M_{\mathrm{dSIRP}} = \begin{pmatrix} \cos(dr)\cos(\mathrm{tilt}) & \cos(dr)\sin(\mathrm{tilt})\sin(\mathrm{tip}) - \cos(\mathrm{tip})\sin(dr) & \cos(dr)\cos(\mathrm{tip})\sin(\mathrm{tilt}) + \sin(dr)\sin(\mathrm{tip}) & dx \\ \cos(\mathrm{tilt})\sin(dr) & \cos(dr)\cos(\mathrm{tip}) + \sin(dr)\sin(\mathrm{tilt})\sin(\mathrm{tip}) & \cos(\mathrm{tip})\sin(dr)\sin(\mathrm{tilt}) - \cos(dr)\sin(\mathrm{tip}) & dy \\ -\sin(\mathrm{tilt}) & \cos(\mathrm{tilt})\sin(\mathrm{tip}) & \cos(\mathrm{tilt})\cos(\mathrm{tip}) & dw \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

References

[MEC] Mark E. Cornell, *Tracker Coordinates: A Proposal*, July 2010.

[Fow] Jim R. Fowler, *Hexapods at the Hobby-Eberly Telescope*, June 2008.

[Ray1] Frank B. Ray, *Tracker Mechanism Kinematics, Servo, and Top-level Software Design for a Semi-transit Telescope with Fixed Spherical Primary Mirror*, HET Technical Report #42, April 4, 1994.

[Ray2] Frank B. Ray, *SST Setting and Tracking Analysis*, HET Technical Report #43, April 5, 1994.